

# ON MOTIVES OF ALGEBRAIC GROUPS ASSOCIATED TO A DIVISION ALGEBRA

EVGENY SHINDER

## 1. INTRODUCTION

In this paper we consider motives and motivic cohomology of algebraic groups  $GL_1(A)$  and  $SL_1(A)$  for a central simple algebra  $A$  of prime degree  $n$  over a field  $F$ . Motivation to study these groups comes from the problems arising in algebraic  $K$ -theory, in particular non-triviality of  $SK_1(A)$  [16], [9].

It is proved by Biglari [4] that motives of *split* reductive algebraic groups such as  $GL_n(F)$  and  $SL_n(F)$  are Tate motives. Furthermore, using higher Chern classes in motivic cohomology constructed by Pushin [11] one can write down explicit direct sum decompositions for the motives of these two groups with integral coefficients. Non-split algebraic groups such as  $GL_1(A)$  and  $SL_1(A)$  are more intricate. We note however that all the complications lie in  $n$ -torsion effects ( $n$  is the degree of  $A$ ): the situation becomes trivial if we make  $n$  invertible.

For  $GL_1(A)$  we follow an idea of Suslin to split the motive  $M(GL_1(A))$  into two pieces: the first piece is a very simple Tate motive, whereas the second piece is a twisted Tate motive  $M$  over  $\mathcal{X}$ , where  $\mathcal{X}$  is the Voevodsky-Chech simplicial scheme associated to the Severi-Brauer variety  $SB(A)$  (Theorem 4.6). We investigate the structure of the latter motive  $M$  using the twisted slice filtration, and compute the second differential in the arising spectral sequence (Theorem 4.7). Using the spectral sequence we compute some lower weight motivic cohomology groups of  $GL_1(A)$  (Corollary 4.9). We also consider the case of degree 2 algebra where one can write explicit decomposition for  $M(GL_1(A))$  (Proposition 4.4).

For  $SL_1(A)$  we only consider algebras  $A$  of degree either 2 or 3. In both of these cases  $SL_1(A)$  admits an explicit smooth compactification  $X_A$  given as a hyperplane section of a generalized Severi-Brauer variety (Proposition 5.1). In the degree 2 case  $X_A$  is a 3-dimensional quadric, whose motive can be computed explicitly (Proposition 5.4). In the degree 3 case  $X_A$  is a hyperplane section of the twisted form of the Grassmannian  $Gr(3, 6)$ . The motives of such hyperplane sections with coefficients in  $\mathbb{Z}/3$  were already considered in [13]. We give a slightly different proof of the decomposition we need with integral (or more precisely, with  $\mathbb{Z}[\frac{1}{2}]$ ) coefficients (Proposition 5.8), using a version of Rost nilpotence theorem that we prove along the way (Proposition 3.9).

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We fix the notation we need.

- $F$  is a perfect field. We assume  $\text{char}(F) \neq 2$  whenever we speak of quaternion algebras and  $\text{char}(F) = 0$  in Section 5.3.
- $A$  is a central simple algebra over  $F$  of degree  $n$ . We assume  $n$  to be prime in Section 4.

- $SB_k(A)$ ,  $k \geq 1$ , is the generalized Severi-Brauer variety introduced in Section 2. In particular  $SB_1(A) = SB(A)$  is the usual Severi-Brauer variety.
- $DM_-^{eff}(F)$  is the Voevodsky triangulated category of motives [17], with the Tate twist  $(*)$  and shift  $[*]$ . We also use  $\{*\} = (*)[2*]$ , especially when working with motives of smooth projective varieties. For example,  $M(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}(1)[2] = \mathbb{Z} \oplus \mathbb{Z}\{1\}$ .
- If  $\mathcal{J}$  is a vector bundle we write  $\mathcal{J}^*$  for the dual bundle.

## 2. VARIETIES ASSOCIATED TO CENTRAL SIMPLE ALGEBRAS

In this section  $A$  is a central simple algebra  $A$  over a field  $F$  of degree  $n$ , i.e. an associative unital algebra of dimension  $n^2$  over  $F$  that has no nontrivial two-sided ideals and such that the center of  $A$  coincides with  $F$ .

According to the Wedderburn theorem,  $A$  is isomorphic to the matrix algebra  $M_n(D)$  over a central division algebra  $D$  over  $F$ .  $A$  is called split if it is isomorphic to  $M_n(F)$ . It is well known that any central simple algebra splits in some finite separable extension of scalars  $E/F$ :

$$A_E = A \otimes_F E \cong M_n(E).$$

By the standard Galois descent arguments the set of isomorphism classes of central simple algebras over  $F$  is in bijection with  $H^1(\text{Gal}(F^{sep}/F), \text{PGL}_n(F^{sep}))$ . Galois descent also implies that  $\det : M_n(F^{sep}) \rightarrow F^{sep}$  and  $\text{tr} : M_n(F^{sep}) \rightarrow F^{sep}$  descend to define the so called reduced norm map  $Nrd : A \rightarrow F$  and the reduced trace map  $\text{Trd} : A \rightarrow F$ .

**Example 2.1.** A quaternion algebra  $\left(\frac{a,b}{F}\right)$  is defined for  $a, b \in F^*$  to be a vector space of dimension 4 with the basis  $1, i, j, k$  and multiplication  $i^2 = a, j^2 = b, ij = -ji = k$  (under the assumption  $\text{char}(F) \neq 2$ ). It follows from the Wedderburn theorem, that  $\left(\frac{a,b}{F}\right)$  either splits or is a division algebra.  $\text{Trd}$  and  $Nrd$  are the usual trace and norm:  $\text{Trd}(x + yi + zj + wk) = 2x$ ,  $Nrd(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2$ .

**2.1. Generalized Severi-Brauer varieties.** The generalized Severi-Brauer variety  $SB_k(A)$  is a closed subvariety in  $\text{Gr}(kn, A)$  representing the functor which associates to a commutative algebra  $R$  over  $F$  the set

$$SB_k(A)(R) = \{\text{right ideals of } A \otimes R \text{ which are projective of rank } kn \text{ over } R\}$$

**Remark 2.2.** 1. Let  $V$  be a vector space of dimension  $n$  over  $F$ , and let  $A$  be a split central simple algebra  $A = \text{End}(V)$ . In this case right ideals in  $A$  of rank  $kn$  have the form  $V^* \otimes U \subset V^* \otimes V \cong \text{End}(V)$ , where  $U$  is a subspace in  $V$  of dimension  $k$ . Therefore we have a canonical identification

$$SB_k(\text{End}(V)) = \text{Gr}(k, V).$$

2. If  $A$  has a right ideal  $I$  of rank  $n$ ,  $A$  is split. Indeed, the right multiplication action  $R_\alpha : I \rightarrow I, a \in A$  satisfies  $R_{\alpha\beta} = R_\beta R_\alpha$ , and the homomorphism

$$R : A \rightarrow \text{End}(I)^{op} = \text{End}(I^*)$$

is an isomorphism by the Schur lemma.

In general  $SB_k(A)$  is a form of  $\text{Gr}(k, V) = \text{Gr}(k, n)$  twisted by the cocycle defining  $A$  and the usual Severi-Brauer variety  $SB(A) = SB_1(A)$  is a twisted form of the projective space  $\mathbb{P}(V) \cong \mathbb{P}^{n-1}$ .

**Example 2.3.** In the case  $A = \begin{pmatrix} a & b \\ & F \end{pmatrix}$ ,  $SB(A)$  is a conic in  $\mathbb{P}^2$  defined by the equation  $x^2 = ay^2 + bz^2$ .

By definition,  $SB_k(A)$  is endowed with a locally free sheaf  $\mathcal{J}_k$  of right  $A$ -modules of rank  $k$ .  $\mathcal{J}_k$  is a subsheaf of  $\mathcal{O}_{SB_k(A)} \otimes A$ .

**Remark 2.4.** In the split case  $A = \text{End}(V)$ ,  $\mathcal{J}_k$  is identified with  $V^* \otimes \xi = \text{Hom}(V, \xi)$  over  $Gr(k, V)$ , where  $\xi$  denotes the tautological rank  $k$  bundle on  $Gr(k, V)$ .

We will write  $\mathcal{J}$  for  $\mathcal{J}_1$  over  $SB(A)$ .  $\mathcal{J}^*$  denotes the dual sheaf.

**Lemma 2.5.** *The sheaf of algebras  $\mathcal{O}_{SB(A)} \otimes A$  is isomorphic to  $\text{End}(\mathcal{J}^*)$ .*

*Proof.* The isomorphism is given by the right action of  $A$  on  $\mathcal{J}$ , as in 2.2, 2.  $\square$

If  $p : E \rightarrow T$  is a vector bundle we will write  $\mathbf{Gr}_T(k, E)$  for the Grassmannian bundle over  $T$  of  $k$ -planes in  $E$ .  $Gr_T(k, E)$  comes equipped with a short exact sequence of vector bundles:

$$0 \rightarrow \xi_E \rightarrow p^*E \rightarrow \mathcal{Q}_E \rightarrow 0.$$

If  $E$  is a trivial bundle over  $T = \text{Spec}(F)$ , we use the notation  $\xi = \xi_E$  and  $\mathcal{Q} = \mathcal{Q}_E$ .

**Proposition 2.6.** *There is a canonical isomorphism of varieties over  $SB(A)$*

$$SB(A) \times SB_k(M_l(A)) \cong \mathbf{Gr}_{SB(A)}(k, \mathcal{J}^{*\oplus l}),$$

where  $\mathcal{J}$  is the tautological sheaf of ideals on  $SB(A)$ .

Furthermore, the tautological bundles  $\xi_{\mathcal{J}^{*\oplus l}}$  and  $\mathcal{Q}_{\mathcal{J}^{*\oplus l}}$  over  $\mathbf{Gr}_{SB(A)}(k, \mathcal{J}^{*\oplus l})$  correspond under this isomorphism to vector bundles over  $SB(A) \times SB_k(M_l(A))$ , which in the split case become identified with  $p_1^*(\mathcal{O}(1)) \otimes p_2^*(\xi)$  and  $p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{Q})$  respectively.

*Proof.* The first assertion follows from Lemma 2.5. To prove the second assertion, consider the split case  $A = \text{End}(V)$ , so we can identify  $\mathcal{J}^*$  with  $V \otimes \mathcal{O}(1)$  by 2.4. Then the isomorphism in question becomes the canonical identification:

$$\mathbf{Gr}_{\mathbb{P}(V)}(k, V^{\oplus l} \otimes \mathcal{O}) \cong \mathbf{Gr}_{\mathbb{P}(V)}(k, V^{\oplus l} \otimes \mathcal{O}(1)),$$

and the claim follows from the lemma below.  $\square$

**Lemma 2.7.** *Let  $E$  be a vector bundle and  $L$  be a line bundle over the same base  $T$ . Then the tautological bundles  $\xi_{E \otimes L}$  and  $\mathcal{Q}_{E \otimes L}$  over  $\mathbf{Gr}_T(k, E \otimes L)$  correspond under the canonical isomorphism*

$$\mathbf{Gr}_T(k, E) \cong \mathbf{Gr}_T(k, E \otimes L)$$

*of varieties over  $T$  to  $p^*L \otimes \xi_E$  and  $p^*L \otimes \mathcal{Q}_E$  respectively ( $p$  is the projection to  $T$ ).*

*Proof.* Consider the pair  $S = (\xi, Gr(k, n))$ .  $S$  admits a  $GL_n(F)$  action defined in an obvious way:

$$(g \in GL_n(F), v \in V \subset F^n) \rightarrow (gv \in gV \subset F^n).$$

Therefore for any cocycle  $\theta \in H_{Zar}^1(T, GL_n(F))$  we can construct a Zarisky locally trivial family  $S_\theta$ . In fact, if  $\theta = \theta_E$  is the cocycle defining  $E$ , then

$$S_{\theta_E} \cong (\xi_E, \mathbf{Gr}_T(k, E)).$$

Now  $\theta_{E \otimes L} = \theta_E \theta_L$ ,  $\theta_L \in H_{Zar}^1(T, \mathbb{G}_m)$ .  $\mathbb{G}_m$  acts on the trivial pair  $S$  by:

$$(\lambda \in \mathbb{G}_m(F), v \in V \subset F^n) \rightarrow (\lambda v \in V \subset F^n).$$

Therefore the pair  $S_{\theta_{E \otimes L}}$  is isomorphic to  $(p^*(L) \otimes \xi_E, \mathbf{Gr}_T(k, E))$ .

The same argument applies to  $\mathcal{Q}_E$ . □

Consider now the case  $SB_n(M_l(A))$ , where  $A$  is a central simple algebra of degree  $n$ .  $SB_n(M_l(A))$  is a twisted form of the Grassmannian  $Gr(n, V^{\oplus l}) = Gr(n, ln)$ .

We will need the following definition.

**Definition 2.8.**  $\alpha_1, \alpha_2, \dots, \alpha_l \in A$  are called independent if there does not exist  $0 \neq \lambda \in A$  with the property  $\alpha_1 \lambda = \alpha_2 \lambda = \dots = \alpha_l \lambda = 0$ .

It is easy to see that  $\alpha_1, \alpha_2, \dots, \alpha_l \in A$  are independent if and only if the *left* ideal  $A\alpha_1 + A\alpha_2 + \dots + A\alpha_l \subset A$  equals  $A$ .

**Remark 2.9.** If  $A$  is a division algebra, then any  $\alpha_1, \alpha_2, \dots, \alpha_l \in A$  are independent unless they are all zero. On the other hand, if  $A$  is a split algebra  $A = \text{End}(V)$ , then  $\alpha_1, \alpha_2, \dots, \alpha_l \in A$  are independent if and only if

$$(\alpha_1, \alpha_2, \dots, \alpha_l) : V \rightarrow V^{\oplus l}$$

is injective.

**Lemma 2.10.** *Let  $A$  be a central simple algebra of rank  $n$  over a field  $F$ .*

1)  $SB_n(M_l(A))$  represents the functor which associates to a commutative  $F$ -algebra  $R$  the set of nondegenerate columns modulo the right action of  $A_R$ :

$$SB_n(M_l(A))(R) = \{(\alpha_1, \alpha_2, \dots, \alpha_l) : \alpha_1, \alpha_2, \dots, \alpha_l \in A_R \text{ independent}\} / A_R^*.$$

2) Vector bundle  $\xi_n^{\oplus n}$  and line bundle  $\Lambda^n \xi_n$  over  $Gr(n, V^{\oplus l})$  canonically descend to  $SB_n(M_l(A))$ .

*Proof.* 1) A right ideal  $R$  of dimension  $ln^2$  in  $M_l(A)$  splits via the action of the idempotents  $e_i = e_{i,i} \in M_l(A)$  as a direct sum of  $l$  isomorphic right  $A$ -submodules  $R_0 \subset A^{\oplus l}$  of dimension  $n^2$ , corresponding to the columns of  $R$ .

Any  $A$ -module is a direct sum of minimal  $A$  modules, and comparing the dimensions we see that in fact  $R_0$  is isomorphic to the trivial right  $A$ -module  $A$ . Let  $\phi : A \rightarrow R_0$  be an isomorphism, and let  $(\alpha_1, \alpha_2, \dots, \alpha_l) = \phi(1)$ . By definition  $\alpha_1, \alpha_2, \dots, \alpha_l$  are independent and any two vectors  $v_1, v_2 \in A^{\oplus l}$  consisting of independent entries generate the same right submodule if and only if  $v_1 = v_2 \lambda$  for some  $\lambda \in A^*$ .

2) The same argument as above applied to the tautological sheaf of ideals  $\mathcal{J}_n$  on  $SB_n(M_l(A))$  shows that this sheaf splits as a direct sum of  $n$  copies of  $A$ -module sheaf, which becomes isomorphic to  $\xi_n^{\oplus n}$  if the algebra splits.  $\Lambda^n \xi_n$  is the line bundle associated with the sheaf of  $A$ -modules  $\xi_n^{\oplus n}$  via the character  $Nrd : GL_1(A) \rightarrow \mathbb{G}_m$ . □

**Remark 2.11.** The usual description of the functor of points for  $Gr(n, ln)$  amounts to choosing  $n$  linearly independent columns of size  $ln$ , forming an  $ln \times n$  matrix of rank  $n$ , which is considered modulo the right action of  $GL_n(F)$ . The lemma says that for  $SB_n(M_l(A))$  the  $n \times n$  blocks of this matrix are rational.

**Remark 2.12.** The second statement of the lemma implies that there is a twisted Plücker embedding

$$SB_n(M_l(A)) \rightarrow \mathbb{P}^{\binom{n}{l}}.$$

To make this map explicit, we pick a basis of the global sections of  $\Lambda^n \xi_n$  consisting of the forms  $H_{\beta_1, \dots, \beta_l}(\alpha_1, \dots, \alpha_l) = \text{Nrd}(\sum_i \beta_i \alpha_i)$ .

For example, in the case of a quaternion algebra  $A$  and  $l = 2$  we can pick the basis to be  $\text{Nrd}(\alpha_1), \text{Nrd}(\alpha_2), \text{Nrd}(\alpha_1 + \alpha_2), \text{Nrd}(\alpha_1 + i\alpha_2), \text{Nrd}(\alpha_1 + j\alpha_2), \text{Nrd}(\alpha_1 + k\alpha_2)$ .

**2.2. Algebraic groups associated to  $A$ .** For a central simple algebra  $A$  over  $F$ , one can define linear algebraic groups  $GL_1(A)$ ,  $SL_1(A)$ . For any  $R$  is a commutative algebra over  $F$  the  $R$ -points of these groups are:

$$GL_1(A)(R) = (A \otimes_F R)^* = \{g \in A \otimes_F R : \text{Nrd}(g) \neq 0\}$$

$$SL_1(A)(R) = \{g \in (A \otimes_F R)^* : \text{Nrd}(g) = 1\}.$$

Note that  $SL_1(A) = \text{Ker}(\text{Nrd} : GL_1(A) \rightarrow \mathbb{G}_m)$  in the category of algebraic groups.  $GL_1(A)$  and  $SL_1(A)$  are forms of  $GL_n(F)$  and  $SL_n(F)$  respectively twisted by the cocycle defining  $A$ .

**Example 2.13.** For the quaternion algebra  $A = \left(\frac{a,b}{F}\right)$ ,  $GL_1(A)$  is an open subscheme in  $\mathbb{A}^4$  defined by  $x^2 - ay^2 - bz^2 + abw^2 \neq 0$ , and  $SL_1(A)$  is a quadric in  $\mathbb{A}^4$  defined by  $x^2 - ay^2 - bz^2 + abw^2 = 1$ .

Let  $E \rightarrow T$  be a vector bundle of rank  $n$  and consider the associated group scheme  $\mathbf{GL}_T(E)$  of local automorphisms of  $E$  over  $T$ . Let  $\alpha_E$  be the tautological automorphism of  $p^*(E)$  where  $p : \mathbf{GL}_T(E) \rightarrow T$  is the projection. Via explicit description of  $K_1$  by Gillet and Grayson [6],  $\alpha_E$  defines an element  $[\alpha_E] \in K_1(\mathbf{GL}_T(E))$ .

This applies in particular to the case of the trivial bundle  $E = V$  over a point, in which case we denote the corresponding element in  $K_1(GL_n(F))$  by  $[\alpha_0]$ .

The following proposition is analogous to 2.6.

**Proposition 2.14.** *There is a canonical isomorphism of varieties over  $SB(A)$*

$$SB(A) \times GL_1(A) \cong \mathbf{GL}_{SB(A)}(J^*),$$

where  $J$  is the tautological sheaf of ideals on  $SB(A)$ .

Furthermore, the tautological class  $[\alpha_{J^*}] \in K_1(\mathbf{GL}(J^*))$  corresponds under this isomorphism to a class in  $K_1(SB(A) \times GL_1(A))$  which in the split case is identified with  $[p_1^*(\mathcal{O}(1))] \cdot [p_2^*(\alpha_0)]$  where the product is the usual product  $K_0 \otimes K_1 \rightarrow K_1$ .

*Proof.* The first assertion follows from Lemma 2.5. To prove the second assertion, consider the split case  $A = \text{End}(V)$ , and we identify  $J^*$  with  $V \otimes \mathcal{O}(1)$  by 2.4. Then the isomorphism in question becomes the canonical identification:

$$\mathbb{P}(V) \times GL_1(\text{End}(V)) = \mathbf{GL}_{\mathbb{P}(V)}(V \otimes \mathcal{O}) = \mathbf{GL}_{\mathbb{P}(V)}(V \otimes \mathcal{O}(1)),$$

and the claim follows from the following lemma. □

**Lemma 2.15.** *Let  $E$  be a vector bundle and  $L$  be a line bundle over the same base  $T$ , which is assumed to be quasiprojective. Then the tautological class  $[\alpha_{E \otimes L}] \in K_1(\mathbf{GL}_T(E \otimes L))$  corresponds to  $[p^*L] \cdot [\alpha_E] \in K_1(\mathbf{GL}_T(E))$  under the canonical isomorphism of group schemes over  $T$  ( $p$  is the projection to  $T$ )*

$$\mathbf{GL}_T(E) \cong \mathbf{GL}_T(E \otimes L).$$

*Proof.* Using the Jouanolu trick, we assume that  $T = \text{Spec}(R)$ . We will use the same letters  $E$  and  $L$  for the  $R$ -projective modules corresponding to bundles  $E$  and  $L$ .

$GL(E)$  is an affine scheme. Let  $S = \Gamma(GL(E), \mathcal{O}_{GL(E)})$ .  $\alpha_E$  is the tautological automorphism of the module  $p^*E$ , and  $\alpha_{E \otimes L} = \alpha_E \otimes id_{p^*L}$  is the automorphism of  $p^*(E \otimes L) = p^*(E) \otimes p^*(L)$ .

It follows from the definition of the product in the K-theory of rings  $K_0(S) \otimes K_1(S) \rightarrow K_1(S)$ , that  $[p^*L] \cdot [\alpha_E] = [\alpha_{E \otimes L}]$ . □

### 3. THE SLICE FILTRATION

We work in the category  $DM_-^{eff}(F)$  of motivic complexes defined by Voevodsky [17]. Recall that  $DM_-^{eff}(F)$  is a tensor triangulated category which admits a covariant monoidal functor

$$M : Sm/F \rightarrow DM_-^{eff}(F),$$

satisfying the usual properties such as Mayer-Vietoris and localization distinguished triangles.

For any smooth variety  $X$  there is a splitting

$$M(X) = \widetilde{M}(X) \oplus \mathbb{Z},$$

where  $\widetilde{M}(X)$  is defined to be the cone of the canonical morphism  $M(X) \rightarrow M(pt) = \mathbb{Z}$ .

Motivic cohomology groups of degree  $p \in \mathbb{Z}$  and weight  $q \geq 0$  are defined to be

$$H^{p,q}(X) := Hom_{DM_-^{eff}(F)}(M(X), \mathbb{Z}(q)[p]),$$

so that distinguished triangles in  $DM_-^{eff}(F)$  become long exact sequences in motivic cohomology of each weight.

We also use reduced motivic cohomology groups

$$\widetilde{H}^{p,q}(X) := Hom_{DM_-^{eff}(F)}(\widetilde{M}(X), \mathbb{Z}(q)[p]).$$

Consider the Čech simplicial scheme  $\mathcal{X} = \check{C}(SB(A))$  [18].  $\mathcal{X}$  is defined such that  $\mathcal{X}_k = SB(A)^{k+1}$  and the face and degeneracy maps are taken to be partial projections and diagonals. The canonical morphism  $M(\mathcal{X}) \rightarrow \mathbb{Z}$  is an isomorphism if  $SB(A)$  has an  $F$ -point (i.e. if algebra  $A$  splits).

We introduce a tensor triangulated category  $DM_-^{eff}(\mathcal{X})$  of motives over  $\mathcal{X}$  as the full subcategory of  $DM_-^{eff}(F)$ , consisting of objects  $M$  satisfying the property that the canonical morphism

$$M \otimes M(\mathcal{X}) \rightarrow M \otimes \mathbb{Z} = M$$

is an isomorphism [19]. Note that  $\mathcal{X}$  is an *embedded* simplicial scheme, which means that  $M(\mathcal{X}) \otimes M(\mathcal{X}) = M(\mathcal{X})$  and so  $M(\mathcal{X})$  is an object in  $DM_{\mathcal{X}}^{eff}$ . We will write  $\mathbb{Z}_{\mathcal{X}}$  for  $M(\mathcal{X})$ .

The full embedding  $DM_{-}^{eff}(\mathcal{X}) \subset DM_{-}^{eff}(F)$  admits a right adjoint functor

$$\Phi : DM_{-}^{eff}(F) \rightarrow DM_{-}^{eff}(\mathcal{X}),$$

which on objects is defined to be

$$\Phi(M) = M \otimes M(\mathcal{X}).$$

**Remark 3.1.** It follows from the adjunction property that for any motive  $M$  in  $DM_{-}^{eff}(\mathcal{X})$ ,  $q \geq 0$ ,  $p \in \mathbb{Z}$

$$H^{p,q}(M, \mathbb{Z}) = Hom_{DM_{-}^{eff}(F)}(M, \mathbb{Z}(q)[p]) \cong Hom_{DM_{-}^{eff}(\mathcal{X})}(M, \mathbb{Z}_{\mathcal{X}}(q)[p]).$$

Let  $DT(\mathcal{X}) \subset DM_{-}^{eff}(\mathcal{X})$  denote the subcategory of effective Tate motives over  $\mathcal{X}$ .

We consider the *slice filtration* on these categories as defined in [19] (see also [7]) for any object  $M$  of  $DM_{-}^{eff}(\mathcal{X})$ . For each  $q \geq 0$  the  $q$ -th term of the slice filtration is given by:

$$\nu_{\mathcal{X}}^{\geq q} M = \underline{Hom}_{DM_{-}^{eff}(F)}(\mathbb{Z}(q), M)(q) \otimes \mathbb{Z}_{\mathcal{X}}.$$

The internal  $Hom$ -object above exists by [17], Proposition 3.2.8.

**Remark 3.2.** It is easy to see using the adjunction property that

$$\underline{Hom}_{DM_{-}^{eff}(F)}(\mathbb{Z}(q), M)(q) \otimes M(\mathcal{X})$$

is in fact isomorphic to

$$\underline{Hom}_{DM_{-}^{eff}(\mathcal{X})}(\mathbb{Z}_{\mathcal{X}}(q), M).$$

On the other hand, if  $M$  is an object in  $DT(\mathcal{X})$ , then  $\underline{Hom}_{DM_{-}^{eff}(\mathcal{X})}(\mathbb{Z}_{\mathcal{X}}(q), M)$  is isomorphic to  $\underline{Hom}_{DT(\mathcal{X})}(\mathbb{Z}_{\mathcal{X}}(q), M)$ , so that for Tate motives our slice filtration coincides with the one from [19].

We define  $\nu_{\mathcal{X}}^q$  as the cone in the distinguished triangle

$$\nu_{\mathcal{X}}^{\geq q+1}(M) \rightarrow \nu_{\mathcal{X}}^{\geq q}(M) \rightarrow \nu_{\mathcal{X}}^q(M) \rightarrow \nu_{\mathcal{X}}^{\geq q+1}(M)[1].$$

One can see that the slice filtration  $\{\nu_{\mathcal{X}}^{\geq q}\}$  is functorial, commutes with extension of scalars and that for each  $k, j \geq 0$  and  $l \in \mathbb{Z}$  satisfies

$$\nu_{\mathcal{X}}^{\geq k+j}(M(j)[l]) = \nu_{\mathcal{X}}^{\geq k}(M)(j)[l]$$

and

$$\nu_{\mathcal{X}}^{\geq k}(M \oplus M') = \nu_{\mathcal{X}}^{\geq k}(M) \oplus \nu_{\mathcal{X}}^{\geq k}(M').$$

**Remark 3.3.** For a split Tate motive  $M = \oplus_{p,q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$  we have

$$\nu_{\mathcal{X}}^{\geq k}(M) = \oplus_{p \geq k, q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$$

and

$$\nu_{\mathcal{X}}^k(M) = \oplus_q \mathbb{Z}_{\mathcal{X}}(k)[q]^{\oplus a_{k,q}}.$$

The following two lemmas provide examples of motives lying in  $DM_{-}^{eff}(\mathcal{X})$  and  $DT(\mathcal{X})$  respectively.

**Lemma 3.4.** *Let  $T$  be a variety over  $F$ .*

1. *If  $T$  is smooth and for each generic point  $\eta$  of  $T$   $A_{F(\eta)}$  is a split algebra then  $M(T)$  lies in  $DM_-^{eff}(\mathcal{X})$ .*
2. *Let  $T \subset S$  be a closed embedding of  $T$  into a smooth variety  $S$ . If for each scheme-theoretic point  $z \in T$   $A_{F(z)}$  is a split algebra then  $M_T(S)$  lies in  $DM_-^{eff}(\mathcal{X})$ .*

*Proof.* To prove the first statement, we need to show that  $M(T) \otimes C = 0$  where  $C = \text{cone}(M(\mathcal{X}) \rightarrow \mathbb{Z})$ . This follows from [18], Lemma 4.5. To prove the second statement, we filter  $T$  by closed subvarieties

$$T_N \subset T_{N-1} \subset \cdots \subset T_1 \subset T_0 = T \subset S$$

where  $T_k \setminus T_{k+1}$  are nonsingular. We prove by the descending induction on  $k$  that  $M_{T_k}(S)$  is an object in  $DM_-^{eff}(\mathcal{X})$ . The base case  $k = N$  follows from the first statement of the Lemma. For the induction step, we use the distinguished triangle in  $DM_-^{eff}(F)$

$$M_{T_k \setminus T_{k+1}}(S \setminus T_{k+1}) \rightarrow M_{T_k}(S) \rightarrow M_{T_{k+1}}(S) \rightarrow M_{T_k \setminus T_{k+1}}(S \setminus T_{k+1})[1]$$

of the triple

$$(S \setminus T_{k+1}, S \setminus T_k) \subset (S, S \setminus T_k) \subset (S, S \setminus T_{k+1}).$$

Since by induction hypothesis and by applying the first part of the Lemma again,  $M_{T_{k+1}}(S)$  and  $M_{T_k \setminus T_{k+1}}(S \setminus T_{k+1})$  lie in  $DM_-^{eff}(\mathcal{X})$ ,  $M_{T_k}(S)$  also lies in  $DM_-^{eff}(\mathcal{X})$ .  $\square$

**Lemma 3.5.** *Let  $M$  be an object in  $DM_-^{eff}(\mathcal{X})$ . Assume that  $M_{F(SB(A))}$  is a split Tate motive of the form  $\oplus_{p,q} \mathbb{Z}(p)[q]^{\oplus a_{p,q}}$ . Then the slice filtration of  $M$  in  $DM_{\mathcal{X}}$  has successive cones which are split Tate motives*

$$\nu_{\mathcal{X}}^p(M) = \oplus_q \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}.$$

*In particular,  $M$  is a mixed Tate motive over  $\mathcal{X}$ .*

For the proof we need the following lemma, which we borrow from [15].

**Lemma 3.6.** *For any  $M$  from  $DM_-^{eff}(F)$  and  $p \in \mathbb{Z}$  the extension of scalars  $H^{p,0}(M) \rightarrow H^{p,0}(M_{F(SB(A))})$  is an isomorphism.*

*Proof.* It is sufficient to prove the statement in the case  $M = M(S)[j]$  where  $S$  is a smooth connected scheme over  $F$ . In this case the homomorphism in question takes the form:

$$H^{p-j,0}(S) \rightarrow H^{p-j,0}(S_{F(SB(A))}),$$

and both groups are equal 0 for  $p \neq j$ .

$S$  is connected, and  $SB(A)$  being geometrically irreducible has separably generated function field  $F(SB(A))$ , hence  $S_{F(SB(A))}$  is connected as well. Therefore if  $p = j$  both cohomology groups in question are isomorphic to  $\mathbb{Z}$  with the map being the identity.  $\square$

Now we can prove Lemma 3.5.



*Proof.* Let  $\nu_{\mathcal{X}}^p M = c_p(M)(p)$ . Then

$$\begin{aligned} & \text{Hom}(\nu_{\mathcal{X}}^p M, \mathbb{Z}_{\mathcal{X}}(p)[q]) \\ &= \text{Hom}(c_p(M), \mathbb{Z}_{\mathcal{X}}[q]) \text{ by the cancellation theorem} \\ &= H^{q,0}(c_p(M), \mathbb{Z}) \text{ by Remark 3.1} \\ &= H^{q,0}(c_p(M_{F(SB(A))}), \mathbb{Z}) \text{ by Lemma 3.6} \\ &= H^{q,0}(\oplus_r \mathbb{Z}[r]^{\oplus a_{p,r}}, \mathbb{Z}) \text{ by Remark 3.3} \\ &= \mathbb{Z}^{\oplus a_{p,q}}. \end{aligned}$$

Therefore there exists a morphism  $\phi_p : \nu_{\mathcal{X}}^p M \rightarrow \oplus_q \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$  such that  $\phi_p$  becomes an isomorphism after scalar extension to  $F(SB(A))$ . This implies that  $\text{cone}(\phi_p)_{F(SB(A))} = 0$ , so that  $\text{cone}(\phi_p) = \text{cone}(\phi_p) \otimes M(\mathcal{X}) = 0$  by [18], Lemma 4.5, and thus  $\phi_p$  is an isomorphism.  $\square$

**Remark 3.7.** As the example of  $M = M(SB(A))$  shows,  $M$  itself is not always a *split* Tate motive. Indeed it is a result of Karpenko [8] that for a division algebra  $A$ ,  $M(SB(A))$  is indecomposable.

**Example 3.8.** Let  $A = \left(\begin{smallmatrix} a & b \\ & F \end{smallmatrix}\right)$ , and let  $M_{a,b} = M(SB(A))$  be the Rost motive. In this case the slice filtration is the distinguished triangle

$$\mathbb{Z}_{\mathcal{X}}(1)[2] \rightarrow M_{a,b} \rightarrow \mathbb{Z}_{\mathcal{X}} \rightarrow \mathbb{Z}_{\mathcal{X}}(1)[3]$$

from [18], Theorem 4.4.

In section 5 we will need the following version of the Rost nilpotence theorem (cf [2], Cor. 8.4 and [12], Cor. 10), which is a corollary of Lemma 3.6 and the existence of the slice filtration:

**Proposition 3.9.** *Let  $M = \bigoplus_{k=0}^n \mathbb{Z}\{i_k\}$  be a pure Tate motive. Let  $f : M(SB(A)) \otimes M \rightarrow M(SB(A)) \otimes M$  be a morphism of motives. If  $f_{F(SB(A))}$  is an isomorphism then  $f$  is an isomorphism.*

*Proof.* Consider the slice filtration on  $M(SB(A)) \otimes M$ . By Lemma 3.5 the slices  $\nu_{\mathcal{X}}^p(M(SB(A)) \otimes M)$  are equal to  $\mathbb{Z}_{\mathcal{X}}\{p\}^{\oplus a_p}$ , for some  $a_p \geq 0$ . The morphisms induced on the slices are given by matrices with coefficients in  $\text{Hom}(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}})$ , and this group is identified with  $\mathbb{Z}$  after extension of scalars to  $F(SB(A))$  by Lemma 3.6.  $\square$

The slice filtration gives rise to an exact couple for each weight  $j$

$$\begin{aligned} E^{p,q} &= H^{p+q}(\nu_{\mathcal{X}}^q(M), \mathbb{Z}(j)), \\ D^{p,q} &= H^{p+q}(M, \nu_{\mathcal{X}}^{\geq q+1}(M), \mathbb{Z}(j)), \\ \dots &\rightarrow D^{p+1,q-1} \rightarrow D^{p,q} \rightarrow E^{p,q} \rightarrow D^{p+2,q-1} \rightarrow \dots \end{aligned}$$

and the corresponding spectral sequence

$$E_2^{p,q} = H^{p+q}(\nu_{\mathcal{X}}^q(M), \mathbb{Z}(j)) \Rightarrow H^{p+q}(M, \mathbb{Z}(j)),$$

with the differential  $d_2 : H^{p+q-1}(\nu_{\mathcal{X}}^{q+1} M, \mathbb{Z}(j)) \rightarrow H^{p+q}(\nu_{\mathcal{X}}^q M, \mathbb{Z}(j))$  induced by a composition of morphisms forming the slice filtration:

$$\nu_{\mathcal{X}}^q(M) \rightarrow \nu_{\mathcal{X}}^{\geq q+1}(M)[1] \rightarrow \nu_{\mathcal{X}}^{q+1}(M)[1].$$

4. THE CASE OF  $GL_1(A)$ 

**4.1. The split case.** We consider the group variety  $GL_n(F)$  over a field  $F$ . To give an explicit description of  $M(GL_n(F))$  we use the higher Chern classes  $c_{1,i}$  for motivic cohomology

$$c_i = c_{1,i} : K_1(X) \rightarrow H^{2i-1,i}(X), \quad i \geq 1.$$

as defined by Pushin [11]. The classes  $c_i$  are functorial, additive, and admit the following product formula: if  $[L] \in K_0(X)$  represents a class of a line bundle  $L$ , with  $\lambda = c_1(L)$  and  $[\alpha] \in K_1(X)$ , then for  $[L] \cdot [\alpha]$  we have

$$c_k([L] \cdot [\alpha]) = \sum_{i=1}^{k-1} (-1)^i \binom{k-1}{i} \lambda^i c_{k-i}(\alpha) =$$

$$c_k(\alpha) - (k-1)\lambda c_{k-1}(\alpha) + \frac{(k-1)(k-2)}{2} \lambda^2 c_{k-2}(\alpha) + \dots + (-1)^{k-1} \lambda^{k-1} c_1(\alpha).$$

For a multi-index

$$I = \{1 \leq i_1 < \dots < i_r \leq n\}$$

we let

$$|I| = i_1 + \dots + i_r$$

$$l(I) = r$$

$$c_I(\alpha) = c_{i_1}(\alpha) \cdot \dots \cdot c_{i_r}(\alpha) \in H^{2|I|-l(I),|I|}(X).$$

**Proposition 4.1.** *The motive  $M(GL_n(F))$  admits the following direct sum decomposition:*

$$M(GL_n(F)) \cong \bigoplus_I \mathbb{Z}(|I|)[2|I| - l(I)],$$

where the morphism

$$M(GL_n(F)) \rightarrow \mathbb{Z}(|I|)[2|I| - l(I)]$$

corresponds to the class

$$c_I(\alpha) \in H^{2|I|-l(I),|I|}(GL_n(F)),$$

$[\alpha]$  is the tautological class in  $K_1(GL_n(F))$  defined in the paragraph preceeding Proposition 2.14.

*Proof.* We define the morphism

$$\phi : M(GL_n(F)) \rightarrow \bigoplus_I \mathbb{Z}(|I|)[2|I| - l(I)]$$

using the classes  $c_I$ . We claim that  $\phi$  is an isomorphism.

First note, that for any reductive split group  $G$  over  $F$  the motive  $M(G)$  is a Tate motive. This is deduced in Biglari's thesis [4] from the Bruhat decomposition of  $G$ .

Since  $M(GL_n(F))$  is a Tate motive, by the Yoneda lemma it is sufficient to check that  $\phi$  induces isomorphism on the motivic cohomology groups.

According to [11], Lemma 13, motivic cohomology of  $GL_n(F)$  is generated freely by the classes  $c_I(\alpha)$  and the statement follows.  $\square$

We also need the relative version of Proposition 4.1.

**Proposition 4.2.** *Let  $E \rightarrow T$  be a vector bundle of rank  $n$ , and let  $\alpha_E$  be the tautological class in  $K_1(\mathbf{GL}(E))$ . The motive  $M(\mathbf{GL}(E))$  admits the following decomposition:*

$$M(\mathbf{GL}(E)) = \bigoplus_I M(T)(|I|)[2|I| - l(I)]$$

where the morphism

$$M(\mathbf{GL}(E)) \rightarrow M(T)(|I|)[2|I| - l(I)]$$

is the composition

$$M(\mathbf{GL}(E)) \rightarrow M(\mathbf{GL}(E)) \otimes M(\mathbf{GL}(E)) \rightarrow M(\mathbf{GL}(E))(|I|)[2|I| - l(I)] \rightarrow M(T)(|I|)[2|I| - l(I)]$$

of multiplication by the class

$$c_I(\alpha_E) \in H^{2|I|-l(I), |I|}(\mathbf{GL}(E)).$$

followed by the canonical projection.

*Proof.* The statement follows from Proposition 4.1 and the Mayer-Vietoris exact triangle.  $\square$

**4.2. The case  $n = 2$ .** Let  $A = \begin{pmatrix} a & b \\ & F \end{pmatrix}$ , and let  $C = SB(A)$ . In this case  $GL_1(A)$  is the complement to  $Q \subset \mathbb{A}^4 - \{0\}$  in  $\mathbb{A}^4 - \{0\}$ , where

$$Q = \{(x, y, z, w) \in \mathbb{A}^4 - \{0\} : x^2 - ay^2 - bz^2 + abw^2 = 0\}.$$

**Lemma 4.3.**  $M(Q) = M(C) \oplus M(C)(2)[3]$ .

*Proof.* First note that the projective quadric  $\{x^2 - ay^2 - bz^2 + abw^2 = 0\} \subset \mathbb{P}^3$  is isomorphic to  $C \times C$ . It follows from Proposition 2.6 that  $C \times C$  is a projective line bundle over  $C$ , therefore

$$M(C \times C) = M(C) \oplus M(C)(1)[2].$$

$Q$  over  $C \times C$  is the complement to the zero section in the line bundle  $\mathcal{O}(-1)$ . We have a distinguished triangle

$$M(C)(1)[1] \oplus M(C)(2)[3] \rightarrow M(Q) \rightarrow M(C) \oplus M(C)(1)[2] \rightarrow M(C)(1)[2] \oplus M(C)(2)[4],$$

and the third morphism is the obvious one and the claim follows.  $\square$

**Proposition 4.4.** *There is a decomposition*

$$M(GL_1(A)) = \mathbb{Z} \oplus M(C)(1)[1] \oplus \mathbb{Z}_{a,b}(3)[4],$$

where  $\mathbb{Z}_{a,b} = \text{cone}(\mathbb{Z}(1)[2] \rightarrow M(C))$ .

*Proof.* Consider the distinguished triangle corresponding to the open embedding  $GL_1(A) \subset \mathbb{A}^4 - \{0\}$ :

$$M(C)(1)[1] \oplus M(C)(3)[4] \rightarrow \widetilde{M}(GL_1(A)) \rightarrow \mathbb{Z}(4)[7] \rightarrow M(C)(1)[2] \oplus M(C)(3)[5].$$

By dimension reasons  $\text{Hom}(\mathbb{Z}(4)[7], M(C)(1)[2]) = 0$ , therefore

$$\widetilde{M}(GL_1(A)) = M(C)(1)[1] \oplus \text{cone}(\mathbb{Z}(4)[7] \rightarrow M(C)(3)[5])[-1].$$

The morphism  $\mathbb{Z}(4)[7] \rightarrow M(C)(3)[5]$  corresponds to a class in  $CH_1(C) = CH^0(C)$  which can be computed after passing to a splitting field. In the split case the morphism in question is the canonical morphism (the one which corresponds to a cycle of degree 1), hence the same holds over  $F$ .  $\square$

**Remark 4.5.** Note that in the split case  $C = \mathbb{P}^1$  and  $\mathbb{Z}_{a,b} = \mathbb{Z}$  so that the we have

$$M(GL_2(F)) = \mathbb{Z} \oplus \mathbb{Z}(1)[1] \oplus \mathbb{Z}(2)[3] \oplus \mathbb{Z}(3)[4]$$

in agreement with Proposition 4.1.

**4.3. The general case.** We assume  $n \geq 3$  is a prime. Let  $Z$  be the complement of  $GL_1(A)$  in  $\mathbb{A}^{n^2} - \{0\}$ , i.e. the subvariety in  $\mathbb{A}^{n^2} - \{0\}$  given by equation  $Nrd_A = 0$ . Let  $M = M_Z(\mathbb{A}^{n^2} - \{0\})[-1]$ , so that there is a distinguished triangle

$$M \rightarrow M(GL_1(A)) \rightarrow M(\mathbb{A}^{n^2} - \{0\}) \rightarrow M[1].$$

**Theorem 4.6.** 1. For  $j < n^2$  and  $p \in \mathbb{Z}$  we have a canonical isomorphism

$$\tilde{H}^{p,j}(GL_1(A)) \rightarrow H^{p,j}(M).$$

2. If  $A$  splits, then we have a decomposition

$$M = \widetilde{M}(GL_1(A)) \oplus \mathbb{Z}(n^2)[2n^2 - 2] = \bigoplus_{I \neq \emptyset} \mathbb{Z}_{\mathcal{X}}(|I|)[2|I| - l(I)] \oplus \mathbb{Z}(n^2)[2n^2 - 2].$$

3.  $M$  is an object in  $DT(\mathcal{X})$  and the slices of the slice filtration are given by:

$$\nu_{\mathcal{X}}^q(M) = \begin{cases} \bigoplus_{|I|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q - l(I)], & 1 \leq q \leq \frac{n(n+1)}{2} \\ \mathbb{Z}_{\mathcal{X}}(n^2)[2n^2 - 2], & q = n^2 \\ 0, & \text{otherwise} \end{cases}$$

*Proof.* Motivic cohomology of  $GL_1(A)$  and that of  $M$  are related via the long exact sequence

$$\tilde{H}^{p,j}(\mathbb{A}^{n^2} - \{0\}) \rightarrow \tilde{H}^{p,j}(GL_1(A)) \rightarrow H^{p,j}(M') \rightarrow \tilde{H}^{p+1,j}(\mathbb{A}^{n^2} - \{0\}),$$

and the first claim follows since

$$\tilde{H}^{p,j}(\mathbb{A}^{n^2} - \{0\}) = H^{p,j}(\mathbb{Z}(n^2)[2n^2 - 1]) = 0$$

for  $j < n^2$  and any  $p \in \mathbb{Z}$ .

If the algebra  $A$  is split, then in the distinguished triangle

$$M \rightarrow \widetilde{M}(GL_n(F)) \rightarrow \widetilde{M}(\mathbb{A}^{n^2} - \{0\}) \rightarrow M[1]$$

the second morphism is zero, since as a simple computation using Proposition 4.1 shows,  $\text{Hom}(\widetilde{M}(GL_n(F)), \widetilde{M}(\mathbb{A}^{n^2} - \{0\})) = 0$ . The triangle splits yielding the first equality in the second claim. The second equality follows from Proposition 4.1.

To prove the third claim note that any point of  $z \in Z$  splits  $A$ :  $A_{F(z)}$  has a non-zero non-invertible element (given by  $z$ ) therefore  $A_{F(z)}$  is not a division algebra, and since we assume that the degree  $n$  of  $A$  is prime,  $A_{F(z)}$  splits. The third claim now follows from Lemmas 3.4, 2 and 3.5.  $\square$

The slice filtration gives rise to a spectral sequence for each weight  $j$

$$E_2^{p,q} = H^{p+q}(\nu_{\mathcal{X}}^q(M), \mathbb{Z}(j)) \Rightarrow H^{p+q}(M, \mathbb{Z}(j)).$$

If we consider the weights  $j < n^2$ , then by Theorem 4.6, 3 the spectral sequence in question actually converges to  $\tilde{H}^{*,j}(GL_1(A))$ .

In the computation of the second differential in the spectral sequence we will need the following isomorphism proved in [10], Proposition 1.3:

$$H^{3,1}(\mathbb{Z}_{\mathcal{X}}) \cong \text{Ker}(\text{res} : H_{\text{et}}^2(F, \mu_n) \rightarrow H_{\text{et}}^2(F(SB(A)), \mu_n)).$$

On the other hand for any field  $H_{\text{et}}^2(F, \mu_l)$  is canonically isomorphic to the  $n$ -torsion of the Brauer group  $Br(E)$ , and the kernel of the restriction map  $Br(F) \rightarrow Br(F(X))$  is generated by the class of algebra  $A$  by the classical Amitsur theorem. Since the period of  $A$  is equal to  $n$  we have a canonical isomorphism

$$H^{3,1}(\mathbb{Z}_{\mathcal{X}}) \cong \mathbb{Z}/n.$$

**Theorem 4.7.** *Let  $1 \leq q \leq \frac{n(n+1)}{2}$ . The second differential  $d_2$  in the slice spectral sequence is induced by the morphism of motives*

$$\partial_q : \nu_{\mathcal{X}}^q(M) = \bigoplus_{|I|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q-l(I)] \rightarrow \nu_{\mathcal{X}}^{q+1}(M)[1] = \bigoplus_{|J|=q+1} \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)].$$

*The morphism*

$$\partial_{I,J} : \mathbb{Z}_{\mathcal{X}}(q)[2q-l(I)] \rightarrow \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)]$$

*corresponding to multi-indices  $I, |I| = q$  and  $J, |J| = q+1$  is zero unless  $l(I) = l(J)$  and the sequence  $J$  is obtained from the sequence  $I$  by increasing one index  $i_t$  by one, in which case  $\partial_{I,J}$  corresponds to the class in  $H^{3,1}(\mathbb{Z}_{\mathcal{X}})$  equal to  $i_t \cdot c \cdot [A]$ , for some integer  $c$  coprime to  $n$  depending only on  $A$ .*

*Proof.* We fix a weight  $q$  and a multi-index  $I$  such that  $|I| = q$ , and let  $r = l(I)$ . Consider the motive  $M(SB(A) \times GL_1(A))$ . According to Proposition 2.14

$$SB(A) \times GL_1(A) = \mathbf{GL}_{SB(A)}(J^*),$$

and Proposition 4.2 implies that  $M(SB(A) \times GL_1(A))$  admits a direct summand  $M(SB(A))(q)[2q-r]$  corresponding to the class  $c_I(\alpha_E)$ . We claim that the composition  $\psi$  defined as

$$\begin{aligned} M(SB(A))(q)[2q-r] &\rightarrow M(\mathbf{GL}_{SB(A)}(J^*)) = \\ &M(GL_1(A) \times SB(A)) \rightarrow M(GL_1(A)) \end{aligned}$$

factors uniquely through  $M \rightarrow M(GL_1(A))$ . Indeed, from the distinguished triangle defining  $M$  we see that it is sufficient to show that

$$\text{Hom}(M(SB(A))(q)[2q-r], M(\mathbb{A}^{n^2} - \{0\})[\epsilon]) = 0,$$

for  $\epsilon = 0, -1$ .

$$\begin{aligned} \text{Hom}(M(SB(A))(q)[2q-r], M(\mathbb{A}^{n^2} - \{0\})[\epsilon]) &= \\ \text{Hom}(M(SB(A))(q)[2q-r], \mathbb{Z}[\epsilon] \oplus \mathbb{Z}(n^2)[2n^2-1+\epsilon]) &= \\ H^{2n^2-1+\epsilon-(2q-r), n^2-q}(SB(A)) &= 0, \end{aligned}$$

since

$$2n^2-1+\epsilon-(2q-r)-(n^2-q) = n^2-q+r-1+\epsilon > \dim(SB(A)) = n-1,$$

where we assume that  $n \geq 3$  and can also assume that  $q < \frac{n(n+1)}{2}$  since otherwise the statement of the theorem is trivial.

The morphism

$$\phi : M(SB(A))(q)[2q - r] \rightarrow M$$

that we have just defined induces a morphism on the slice filtrations of the source and target motives. By Lemma 3.5 we have

$$\nu_{\mathcal{X}}^k(M(SB(A))(q)[2q - r]) = \mathbb{Z}_{\mathcal{X}}(k)[2k - r]$$

for  $q \leq k \leq n - 1 + q$ .

On the other hand by Theorem 4.6 we have

$$\nu_{\mathcal{X}}^k(M) = \bigoplus_{|J|=k} \mathbb{Z}_{\mathcal{X}}(k)[2k - l(J)].$$

For each  $J$  with  $|J| = k$  we let  $\nu_{\mathcal{X}}^k(\phi) = \bigoplus \nu_{\mathcal{X}}^k(\phi)_J$ ,

$$\nu_{\mathcal{X}}^k(\phi)_J : \mathbb{Z}_{\mathcal{X}}(k)[2k - r] \rightarrow \mathbb{Z}_{\mathcal{X}}(k)[2k - l(J)].$$

Each  $\nu_{\mathcal{X}}^k(\phi)_J$  is an element in

$$\text{Hom}(\mathbb{Z}_{\mathcal{X}}(k)[2k - r], \mathbb{Z}_{\mathcal{X}}(k)[2k - l(J)]) =$$

$$\text{Hom}(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}}[r - l(J)]) = H^{r-l(J), 0}(\mathbb{Z}_{\mathcal{X}}).$$

By Lemma 3.6 the latter group is isomorphic to  $\mathbb{Z}$  when  $l(J) = r$  and is zero otherwise.

For the computation of the differential in the spectral sequence we restrict our attention to the cases  $k = q$ ,  $k = q + 1$ .

**Lemma 4.8.** 1.  $\nu_{\mathcal{X}}^q(\phi)_J = 1$  for  $J = I$  and is zero if  $J \neq I$ .

2.  $\nu_{\mathcal{X}}^{q+1}(\phi)_J = i_t$  if the sequence  $J$  is obtained from  $I$  by increasing an index  $i_t$  by one, and is zero otherwise.

Using the Lemma we finish the proof of the theorem. We have the commutative diagram of the connecting morphisms in the slice filtrations:

$$\begin{array}{ccc} \mathbb{Z}_{\mathcal{X}}(q)[2q - r] & \xrightarrow{\quad \bar{\partial} \quad} & \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-r] \\ \nu_{\mathcal{X}}^q(\phi) \downarrow & & \nu_{\mathcal{X}}^{q+1}(\phi) \downarrow \\ \bigoplus_{|J|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q - l(J)] & \xrightarrow{\quad \partial_q \quad} & \bigoplus_{|J|=q+1} \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)] \end{array}$$

From the first part of the Lemma it follows that the left vertical map is the canonical embedding corresponding to  $J = I$ . A diagram chase shows that

$$\partial_{I,J} = \nu_{\mathcal{X}}^{q+1}(\phi)_J \circ \bar{\partial}.$$

It is proved by Suslin in [15] that  $\bar{\partial} \in H^{3,1}(\mathbb{Z}_{\mathcal{X}})$  is equal to  $c[A]$ , for  $c$  coprime to  $n$ . By the second part of the Lemma, we obtain the description of the differential.

It remains to prove the Lemma above. According to Lemma 3.6, integers  $\nu_{\mathcal{X}}^q(\phi)_J$  and  $\nu_{\mathcal{X}}^{q+1}(\phi)_J$  do not change under the extension of scalars to the field  $F(SB(A))$ . Therefore we may assume that  $A$  is split.

In this case for any  $1 \leq q \leq \frac{n(n+1)}{2}$  we have  $\nu_{\mathcal{X}}^q(M) = \nu_{\mathcal{X}}^q(GL_n(F))$  by Theorem 4.6, 2 and thus  $\nu_{\mathcal{X}}^q(\phi)$  is identified with  $\nu_{\mathcal{X}}^q(\psi)$ .

The morphism  $\psi$  has the form:

$$\psi : M(\mathbb{P}(V))(q)[2q-r] \rightarrow M(\mathbf{GL}_{\mathbb{P}(V)}(J^*)) = M(\mathbb{P}(V) \times GL_n(F)) \rightarrow M(GL_n(F)).$$

For each  $q \leq k \leq q+n-1$  we have

$$\nu_{\mathcal{X}}^k(\psi) : \mathbb{Z}(k)[2k-r] \rightarrow \bigoplus_{|J|=k} \mathbb{Z}(k)[2k-l(J)].$$

$\nu_{\mathcal{X}}^k(\psi)_J$  can be non-zero only for  $J$  with  $l(J) = r$ , and for such  $J$ , we have

$$\nu_{\mathcal{X}}^k(\psi)_J = \psi^*(c_J(\alpha_0)) \in CH^{k-q}(\mathbb{P}(V)).$$

$\psi^*$  above is the morphism induced on motivic cohomology:

$$\psi^* : H^{*,*}(GL_n(F)) \rightarrow H^{*,*(2q-r),*-q}(\mathbb{P}(V)).$$

By Proposition 4.2 motivic cohomology  $H^{*,*}(\mathbf{GL}_{\mathbb{P}(V)}(J^*))$  is a free module over  $H^{*,*}(\mathbb{P}(V))$  with a basis  $c_J(\alpha_{J^*})$ . We also have another basis  $c_J(p_2^*\alpha_0)$ , and the map  $\psi^*$  above can be described in terms of these bases as follows:  $\psi^*(c_J(\alpha_0))$  is the coefficient of  $\lambda^{|J|-q}c_J(\alpha_{J^*})$  in  $c_J(p_2^*\alpha_0)$ , where  $\lambda = [\mathcal{O}(1)] \in CH^1(\mathbb{P}(V))$ .

By Proposition 2.14  $[p_2^*(\alpha_0)] = [p_1^*(\mathcal{O}(-1))] \cdot [\alpha]$  and by the properties of the higher Chern classes listed before Proposition 4.1 we compute:

$$\begin{aligned} c_{j_1, \dots, j_r}(p_2^*(\alpha_0)) &= \prod_{t=1}^r c_{j_t}(p_2^*(\alpha_0)) = \\ &= \prod_{t=1}^r (c_{j_t}(\alpha) + (j_t - 1)\lambda c_{j_t-1}(\alpha) + \dots) = \\ &= c_{j_1, \dots, j_r}(\alpha) + \sum_{t=1}^r (j_t - 1)\lambda c_{j_1, \dots, j_t-1, \dots, j_r}(\alpha) + \dots \end{aligned}$$

Taking  $|J| = q$ ,  $|J| = q+1$  in the formula above finishes the proof.  $\square$

**Corollary 4.9.** *The extension of scalars to a splitting field of  $A$  identifies the weight 1 and 2 motivic cohomology of  $GL_1(A)$  with:*

$$\begin{aligned} \tilde{H}^{p,1}(GL_1(A)) &= \begin{cases} \mathbb{Z}, & p = 1 \\ 0, & \text{otherwise} \end{cases} \\ \tilde{H}^{p,2}(GL_1(A)) &= \begin{cases} F^*, & p = 2 \\ n\mathbb{Z}, & p = 3 \\ 0, & \text{otherwise} \end{cases} \\ \tilde{H}^{p,3}(GL_1(A)) &= \begin{cases} H^{0,2}(F), & p = 1 \\ H^{1,2}(F), & p = 2 \\ H^{2,2}(F), & p = 3 \\ \mathbb{Z} \oplus (F^*)^n, & p = 4 \\ n\mathbb{Z}, & p = 5 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

*Proof.* In weight  $j$  the spectral sequence has nonzero terms

$$E_2^{p,q} = H^{p+q}(\nu_{\mathcal{X}}^q(M), \mathbb{Z}(j))$$

only for  $0 < q \leq j$ . Let us now consider the weights  $j = 1, 2, 3$ . In these weights the spectral sequence converges to  $\tilde{H}^{*,j}(GL_1(A))$  by theorem 4.6, 1. The first three slices of the slice filtration are given by:

$$\begin{aligned}\nu_{\mathcal{X}}^1(M) &= \mathbb{Z}_{\mathcal{X}}(1)[1] \\ \nu_{\mathcal{X}}^2(M) &= \mathbb{Z}_{\mathcal{X}}(2)[3] \\ \nu_{\mathcal{X}}^3(M) &= \mathbb{Z}_{\mathcal{X}}(3)[4] \oplus \mathbb{Z}_{\mathcal{X}}(3)[5].\end{aligned}$$

We need the following result on motivic cohomology of  $\mathcal{X}$ :

$$H^{p,0}(\mathcal{X}) = \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & \text{otherwise} \end{cases}$$

which follows from Lemma 3.6, and

$$\begin{aligned}H^{p,1}(\mathcal{X}) &= \begin{cases} F^*, & p = 1 \\ \mathbb{Z}/n \cdot [A], & p = 3 \\ 0, & \text{otherwise} \end{cases} \\ H^{p,2}(\mathcal{X}) &= \begin{cases} H^{p,2}(F), & p \leq 2 \\ K_1(F)/n \cdot [A], & p = 4 \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

These formulas follow easily from the results on motivic cohomology of more general simplicial schemes  $\mathcal{X}_{\theta}$  proved in [10].

In the weight  $j = 1$  the slice spectral sequence consists of one row which contains a unique non-zero term  $E_2^{0,1} = H^{0,0}(\mathcal{X}) = \mathbb{Z}$ , hence we get the isomorphism

$$\tilde{H}^{1,1}(GL_1(A)) = \mathbb{Z}$$

and the other reduced cohomology groups of  $GL_1(A)$  of weight 1 vanish.

In the weight  $j = 2$  we have two nonzero rows:

$$\begin{aligned}E_2^{p,1} &= H^{p+1,2}(\mathcal{X}(1)[1]) = H^{p,1}(\mathcal{X}) \\ E_2^{p,2} &= H^{p+2,2}(\mathcal{X}(2)[3]) = H^{p-1,0}(\mathcal{X})\end{aligned}$$

$$\begin{array}{ccccccc} 0 & & \mathbb{Z} & & 0 & & 0 \\ & & \searrow & & & & \\ & & & d^2 & & & \\ 0 & & F^* & & 0 & & \mathbb{Z}/n \end{array}$$

$$0 \quad 0 \quad 0 \quad 0$$

and the differential  $d_2$  is multiplication by  $c$  which is coprime to  $n$ , thus

$$\tilde{H}^{2,2}(GL_1(A)) = F^*$$

$$\tilde{H}^{3,2}(GL_1(A)) = n\mathbb{Z}$$

and the other reduced cohomology groups of  $GL_1(A)$  of weight 2 vanish.



In the weight  $j = 3$  we have three nonzero rows:

$$\begin{aligned} E_2^{p,1} &= H^{p+1,3}(\mathcal{X}(1)[1]) = H^{p,2}(\mathcal{X}) \\ E_2^{p,2} &= H^{p+2,3}(\mathcal{X}(2)[3]) = H^{p-1,1}(\mathcal{X}) \\ E_2^{p,3} &= H^{p+3,3}(\mathcal{X}(3)[4] \oplus \mathcal{X}(3)[5]) = H^{p-1,0}(\mathcal{X}) \oplus H^{p-2,0}(\mathcal{X}). \end{aligned}$$

$$\begin{array}{ccccccc} 0 & \mathbb{Z} & \mathbb{Z} & 0 & 0 & & \\ & & & \searrow d^2 & & & \\ 0 & 0 & F^* & 0 & H^{3,1}(\mathcal{X}) & & \\ & & & \searrow d^2 & & & \\ H^{0,2}(F) & H^{1,2}(F) & H^{2,2}(F) & 0 & H^{4,2}(\mathcal{X}) & & \\ & 0 & 0 & 0 & 0 & 0 & \end{array}$$

Both non-zero differentials are surjective with kernels consisting of elements divisible by  $n$ . There are no higher differentials by dimension reasons.  $\square$

## 5. THE CASE OF $SL_1(A)$

In this section we investigate the motive of  $SL_1(A)$  in the case of a division algebra  $A$  of degree 2 or 3 by looking at the smooth compactification of  $SL_1(A)$ . The compactification in question arises as a hyperplane section of the generalized Severi-Brauer variety associated to  $M_2(A)$ .

**5.1. Hyperplane sections of the generalized Severi-Brauer varieties.** Let  $A$  be a division algebra of degree  $n$  and let  $X_A$  be the closed subvariety of  $SB_n(M_2(A))$  given by equation

$$Nrd(\alpha_1) = Nrd(\alpha_2).$$

$X_A$  is a hyperplane section of  $SB_n(M_2(A))$  with respect to the Plücker embedding 2.12.

On the open dense subscheme of  $X_A$  where  $Nrd(\alpha_1) \neq 0$  we have

$$[\alpha_1, \alpha_2] = [1, \alpha_1^{-1}\alpha_2] = [1, \alpha],$$

with the condition  $Nrd(\alpha) = 1$ . Hence this open subscheme is isomorphic to  $SL_1(A)$ .

**Proposition 5.1.**  *$X$  is smooth if the degree  $n = 2$  or 3.*

*Proof.* We concentrate on the case  $n = 3$ , the case  $n = 2$  being similar. It suffices to consider the case when  $A$  splits. In this case, according to 2.2,  $SB_3(M_2(A))$  is isomorphic to  $Gr(3, 6)$ , and  $Nrd$  is identified with  $\det$ . We identify points of  $Gr(3, 6)$  with  $6 \times 3$  matrices of rank 3 modulo  $GL_3(F)$  action on the columns. We cover  $Gr(3, 6)$  by open charts where the three given rows are linearly independent.

$X_A$  intersects the open chart where the first three (or the last three) rows are linearly independent, in a variety isomorphic to  $SL_3(F)$ , which is obviously smooth.

All other cases will look essentially like this:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{a_{31}}{0} & \frac{a_{32}}{0} & \frac{a_{33}}{1} \\ a_{51} & a_{52} & a_{53} \\ a_{61} & a_{62} & a_{63} \end{pmatrix},$$

in which case the equation defining  $X_A$  becomes

$$a_{33} = a_{51}a_{62} - a_{52}a_{61},$$

so that in these charts  $X_A$  is also smooth.  $\square$

**Remark 5.2.** It is clear from the proof of the proposition that  $X_A$  will not be smooth if  $n > 3$ . Indeed, in this case the equation defining  $X_A$  in some of the charts will be polynomials that do not contain constant or linear terms in  $a_{ij}$ .

**5.2. The case  $n = 2$ .** Let  $A = \left(\begin{smallmatrix} a & b \\ & F \end{smallmatrix}\right)$  be the quaternion algebra, and  $X = X_A$ . As indicated in Remark 2.12 we can pick coordinates such that the Plücker embedding  $SB_2(M_2(A)) \rightarrow \mathbb{P}^5$  sends  $[\alpha_1, \alpha_2]$  to

$$[Nrd(\alpha_1), Nrd(\alpha_2), Nrd(\alpha_1 + \alpha_2), Nrd(\alpha_1 + i\alpha_2), Nrd(\alpha_1 + j\alpha_2), Nrd(\alpha_1 + k\alpha_2)].$$

Note that  $SB_2(M_2(A))$  is a hypersurface in  $\mathbb{P}^5$ . We can change a coordinate system, so that the equation defining  $SB_2(M_2(A))$  becomes particularly simple.

**Proposition 5.3.** *Let*

$$t_1 = Nrd(\alpha_1)$$

$$t_2 = Nrd(\alpha_2)$$

and

$$\begin{aligned} u_1 &= \frac{1}{2}(Nrd(\alpha_1 + \alpha_2) - Nrd(\alpha_1) - Nrd(\alpha_2)) \\ u_2 &= -\frac{1}{2a}(Nrd(\alpha_1 + i\alpha_2) - Nrd(\alpha_1) - Nrd(i\alpha_2)) \\ u_3 &= -\frac{1}{2b}(Nrd(\alpha_1 + j\alpha_2) - Nrd(\alpha_1) - Nrd(j\alpha_2)) \\ u_4 &= \frac{1}{2ab}(Nrd(\alpha_1 + k\alpha_2) - Nrd(\alpha_1) - Nrd(k\alpha_2)). \end{aligned}$$

Then  $SB_2(M_2(A))$  is defined by the equation

$$t_1 t_2 = u_1^2 - a u_2^2 - b u_3^2 + a b u_4^2.$$

*Proof.* First note that the identity

$$Nrd(\alpha + \beta) = Nrd(\alpha) + Nrd(\beta) + Tr(\alpha \bar{\beta}).$$

implies that

$$\begin{aligned} u_1 &= \frac{1}{2}Tr(\alpha_1 \bar{\alpha}_2) \\ u_2 &= -\frac{1}{2a}Tr(\alpha_1 \overline{i\alpha_2}) = \frac{1}{2a}Tr(\alpha_1 \bar{\alpha}_2 i) \\ u_3 &= -\frac{1}{2b}Tr(\alpha_1 \overline{j\alpha_2}) = \frac{1}{2b}Tr(\alpha_1 \bar{\alpha}_2 j) \\ u_4 &= \frac{1}{2ab}Tr(\alpha_1 \overline{k\alpha_2}) = -\frac{1}{2ab}Tr(\alpha_1 \bar{\alpha}_2 k). \end{aligned}$$

Now,

$$\begin{aligned}\alpha_1 \bar{\alpha}_2 &= \frac{1}{2} \text{Tr}(\alpha_1 \bar{\alpha}_2) + \frac{1}{2a} \text{Tr}(\alpha_1 \bar{\alpha}_2 i) i + \frac{1}{2b} \text{Tr}(\alpha_1 \bar{\alpha}_2 j) j - \frac{1}{2ab} \text{Tr}(\alpha_1 \bar{\alpha}_2 k) k = \\ &u_1 + u_2 i + u_3 j + u_4 k,\end{aligned}$$

so that

$$t_1 t_2 = \text{Nrd}(\alpha_1) \text{Nrd}(\alpha_2) = \text{Nrd}(\alpha_1) \text{Nrd}(\bar{\alpha}_2) =$$

$$\text{Nrd}(\alpha_1 \bar{\alpha}_2) = \text{Nrd}(u_1 + u_2 i + u_3 j + u_4 k) = u_1^2 - au_2^2 - bu_3^2 + abu_4^2.$$

It follows that  $SB_2(M_2(A))$  is contained in the quadric

$$t_1 t_2 = u_1^2 - au_2^2 - bu_3^2 + abu_4^2.$$

Since both varieties are irreducible and of the same dimension, they must coincide.  $\square$

Let  $C = SB(A)$ .

**Proposition 5.4.**  $M(SL_1(A)) = \mathbb{Z} \oplus \mathbb{Z}_{a,b}(2)[3]$ , where  $\mathbb{Z}_{a,b} = \text{cone}(\mathbb{Z}(1)[2] \rightarrow M(C))$ .

*Proof.* The equation defining  $X$  in  $\mathbb{P}^4$  is

$$t^2 = u_1^2 - au_2^2 - bu_3^2 + abu_4^2,$$

which is the projective closure of the affine quadric  $SL_1(A) \subset \mathbb{A}^4$

$$u_1^2 - au_2^2 - bu_3^2 + abu_4^2 = 1.$$

Quadratic form  $\langle 1, -a, -b, ab, -1 \rangle$  is equivalent to the sum of a hyperbolic plane and  $\langle 1, -a, -b \rangle$ . It follows from the work of Rost [12] that

$$M(X) = \mathbb{Z} \oplus M(C)(1)[2] \oplus \mathbb{Z}(3)[6].$$

The complement to  $SL_1(A)$  inside  $X$  is the smooth Pfister quadric

$$u_1^2 - au_2^2 - bu_3^2 + abu_4^2 = 0,$$

isomorphic to  $C \times C$ .

We have the localization distinguished triangle

$$\widetilde{M}(SL_1(A)) \rightarrow M(C)(1)[2] \oplus \mathbb{Z}(3)[6] \rightarrow M(C)(1)[2] \oplus M(C)(2)[4] \rightarrow M(SL_1(A))[1].$$

One can see that

$$\begin{aligned}\widetilde{M}(SL_1(A)) &= \text{cone}(\mathbb{Z}(3)[6] \rightarrow M(C)(2)[4])[-1] = \\ &\text{cone}(\mathbb{Z}(1)[2] \rightarrow M(C)(2)[3]) = \mathbb{Z}_{a,b}(2)[3].\end{aligned}$$

$\square$

**5.3. The case  $n = 3$ .** In this section we consider the motive of  $X_A$  for an algebra of degree 3.

Let  $T$  be a smooth projective variety of dimension  $d$  over  $F$ . Consider a Tate motive  $\bigoplus_{i=0}^d \mathbb{Z}\{i\}^{\oplus k_i}$ , which we require to be  $d$ -self-dual:  $k_i = k_{d-i}$  for all  $i$ . Let  $\phi$  be a morphism of motives:

$$\phi : \bigoplus_{i=0}^d \mathbb{Z}\{i\}^{\oplus k_i} \rightarrow M(T).$$

We can consider the dual morphism

$$M(T)(-d)[-2d] = M(T)^* \rightarrow \bigoplus_{i=0}^d \mathbb{Z}\{-i\}^{\oplus k_i},$$

and its twist

$$\phi^t : M(T) \rightarrow \bigoplus_{i=0}^d \mathbb{Z}\{d-i\}^{\oplus k_i} = \bigoplus_{i=0}^d \mathbb{Z}\{i\}^{\oplus k_i}.$$

**Remark 5.5.** We have canonical identifications:

$$CH_i(T) = \text{Hom}_{DM_{-}^{eff}(F)}(\mathbb{Z}\{i\}, M(T))$$

$$CH^i(T) = \text{Hom}_{DM_{-}^{eff}(F)}(M(T), \mathbb{Z}\{i\}).$$

From this point of view both  $\phi$  and  $\phi^t$  correspond to the same set of elements  $\{\alpha_{ij} \in CH_i(T) = CH^{d-i}(T), j = 1 \dots k_i\}_{i=0}^d$

**Lemma 5.6.** *The composition  $\phi^t \circ \phi$  is an isomorphism if and only if the Gram matrix for the intersection product of  $\{\alpha_{ij}\}$  has an invertible determinant.*

*Proof.* First recall, that

$$\text{Hom}(\mathbb{Z}\{n\}, \mathbb{Z}\{m\}) = \begin{cases} \mathbb{Z}, & n = m \\ 0, & \text{otherwise} \end{cases}$$

This reduces the general case to the following two special cases of the  $d$ -self-dual Tate motive:  $\mathbb{Z}\{i\}^{\oplus k} \oplus \mathbb{Z}\{d-i\}^{\oplus k}$ ,  $i \neq d/2$  and  $\mathbb{Z}\{d/2\}^{\oplus k}$  (if  $d$  is even). In the first case,  $\phi : \mathbb{Z}\{i\}^{\oplus k} \oplus \mathbb{Z}\{d-i\}^{\oplus k} \rightarrow M(T)$  is determined by the classes  $\alpha_1, \dots, \alpha_k \in CH_i(T)$  and  $\beta_1, \dots, \beta_k \in CH^i(T)$ . Let  $A$  be the  $k \times k$  matrix with entries  $\alpha_i \cdot \beta_j \in \mathbb{Z}$ . Then the matrix of  $\phi^t \circ \phi$  is the  $2k \times 2k$  block matrix

$$\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},$$

which is invertible if and only  $A$  is invertible.

The case of  $\mathbb{Z}\{d/2\}^{\oplus k}$  is similar: the morphism  $\phi : \mathbb{Z}\{d/2\}^{\oplus k} \rightarrow M(T)$  is determined by the collection of classes  $\alpha_1, \dots, \alpha_k \in CH^{d/2}(T)$ , and the matrix of  $\phi^t \circ \phi$  is equal to the Gram matrix of  $\{\alpha_i\}$ . □

Let  $A$  be an algebra of degree 3 over a field  $F$  of characteristic 0. We set  $X = X_A$ . We start with the split case  $A = \text{End}(V)$  when  $SB_3(M_2(A))$  is identified with  $Gr(3, V \oplus V) = Gr(3, 6)$ .

Recall that  $Gr(3, 6)$  is a cellular variety and the Chow groups  $CH^*(Gr(3, 6))$  are freely generated by Schubert cells  $\Delta_\lambda$  corresponding to Young diagrams  $\lambda$  with 3 rows and 3 columns [5]. We will be using the so-called Pieri formula [5] to compute

the product of any  $\Delta_\lambda$  with the generator  $\Delta_{(1)} = \square \in CH^1(Gr(3, 6))$ . The result of multiplication is the sum of all  $\Delta_{\lambda'}$  for partitions  $\lambda'$  which can be obtained from  $\lambda$  by adding one box. For example,

$$\square \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}.$$

Following Semenov's paper [13], in the split case  $\mathbb{G}_m$  is acting on  $X$  with finitely many fixed points, and using the results of Bialinicki-Birula [3] we conclude that in split case  $X$  is also cellular, in particular the motive of  $X$  is a Tate motive and all cohomology classes on  $X$  are algebraic. It follows now from the Weak Lefschetz theorem that the natural pull-back and push-forward maps

$$CH^i(Gr(3, 6)) \rightarrow CH^i(X), \quad 0 \leq i \leq 3$$

and

$$CH^i(X) \rightarrow CH^{i+1}(Gr(3, 6)), \quad 5 \leq i \leq 8$$

are isomorphisms.

The map in the middle codimension

$$CH^4(Gr(3, 6)) \rightarrow CH^4(X)$$

is injective with cokernel of rank 1.

Under these identifications we will consider Schubert classes corresponding to partitions  $\lambda$  with  $|\lambda| = i$  as elements of  $CH^i(X)$  for  $i \leq 4$  and as elements of  $CH^{i-1}(X)$  for  $i \geq 4$ . For example

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_X := i^*(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_{Gr}) \in CH^4(X)$$

whereas

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}_X := i_*^{-1}(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{Gr}) \in CH^5(X).$$

For these classes the Pieri formula holds as above with the exception of codimension 4 case, where one has to apply it twice, for example

$$\begin{aligned} \square_X \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_X &= i_*^{-1} i_*(\square_X \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_X) = i_*^{-1} i_* i^*(\square_{Gr} \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{Gr}) = \\ i_*^{-1}(\square_{Gr}^2 \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{Gr}) &= i_*^{-1}(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}_{Gr} + 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{Gr} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_{Gr}) = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_X + 2 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}_X + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}_X \end{aligned}$$

Now consider the general (non-split) case. Let  $E$  be a splitting field of the algebra  $A/F$ . For a variety  $T/F$  we call a class  $\alpha \in CH^*(T_E)$  *rational* if it lies in the image of the extension of scalars map  $CH^*(T) \rightarrow CH^*(T_E)$ .

**Proposition 5.7.** *The following classes are rational:*

1.  $\square \in CH^1(Gr(3, 6)_E), \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \in CH^4(Gr(3, 6)_E)$
2.  $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} + H \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + H^2 \square \in CH^3(\mathbb{P}_E^2 \times Gr(3, 6)_E)$ , where  $H$  is the hyperplane section class on  $\mathbb{P}_E^2$ .
3. All classes divisible by 3 in  $CH^*(\mathbb{P}_E^2), CH^*(Gr(3, 6)_E)$  or  $CH^*(X_E)$

*Proof.* 1. For  $i = 1, 2$  consider the subvarieties  $Z_i \subset SB_3(M_2(A))$  defined by equations

$$rk(\alpha_i) \leq i.$$

In fact, extending scalars to the splitting field  $E$ ,  $Z_1$  and  $Z_2$  are the closures of the open Schubert cells

$$\begin{pmatrix} 1 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 0 \\ \hline 0 & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}$$

respectively. Therefore the class of  $Z_1$  is  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ , and the class of  $Z_2$  is  $\square$ .

2. According to Proposition 2.6 the bundle  $p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{Q})$  is rational. For any rank 3 bundle  $E$  and line bundle  $L$

$$c_3(L \otimes E) = c_3(E) + c_1(L)c_2(E) + c_1(L)^2c_1(E),$$

therefore

$$c_3(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{Q})) = \square\square\square + H\square\square + H^2\square$$

and thus this cycle is rational.

3. This follows from the transfer argument, since  $A$  has a splitting field of degree 3.

□

**Proposition 5.8.** *Consider the following five elements*

$$\alpha_1 = \square\square\square + H\square\square + H^2\square \in CH^3(\mathbb{P}_E^2 \times X_E)$$

$$\alpha_2 = \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + H(\square\square\square + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) + H^2(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \end{smallmatrix}) \in CH^4(\mathbb{P}_E^2 \times X_E)$$

$$\alpha_3 = (\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} - \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}) + H(-\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) + H^2(\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} - \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + \begin{smallmatrix} \square \\ \square \end{smallmatrix}) \in CH^5(\mathbb{P}_E^2 \times X_E)$$

$$\alpha_4 = -\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + H(-\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} - \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) + H^2(-\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \in CH^6(\mathbb{P}_E^2 \times X_E)$$

$$\alpha_5 = -\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + H(-\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}) + H^2(-\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} + \begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix} - \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}) \in CH^7(\mathbb{P}_E^2 \times X_E)$$

Then

1.  $\alpha_1, \dots, \alpha_5$  are rational cycles, therefore each  $\alpha_i$  defines a morphism

$$M(SB(A))\{6-i\} \rightarrow M(X)$$

2. Cycles  $\alpha_i, i = 1 \dots 5$  together with the canonical maps  $\mathbb{Z} \rightarrow M(X)$  and  $\mathbb{Z}\{8\} \rightarrow M(X)$  define a morphism

$$\phi : \mathbb{Z} \oplus \bigoplus_{i=1}^5 M(SB(A))\{i\} \oplus \mathbb{Z}\{8\} \rightarrow M(X)$$

which is an embedding of a direct summand if we consider a coefficient ring where 2 is invertible. The complementary direct summand is a form of  $\mathbb{Z}\{4\}$

*Proof.* The class  $\alpha_1$  is rational by Proposition 5.7. Now using Pieri formula from section 5.3 one can check that for all  $i$ ,

$$\alpha_{i+1} \equiv \square \cdot \alpha_i \mod 3,$$

which again by Proposition 5.7 implies that all  $\alpha_i$  are rational.

To prove the second claim, consider the dual map

$$\phi^t : M(X) \rightarrow \mathbb{Z} \oplus \bigoplus_{i=1}^5 M(SB(A))\{i\} \oplus \mathbb{Z}\{8\}.$$

It is sufficient to check that the composition  $\phi^t \circ \phi$  is an isomorphism. First note that by dimension reasons there is no morphisms between  $\mathbb{Z} \oplus \mathbb{Z}\{8\}$  and  $\bigoplus_{i=1}^5 M(SB(A))\{i\}$ . Therefore what we have to check is that the restrictions of  $f^t \circ f$  to  $\mathbb{Z} \oplus \mathbb{Z}\{8\}$  and  $\bigoplus_{i=1}^5 M(SB(A))(i)$  are isomorphisms.

The restriction of  $\phi^t \circ \phi$  to  $\mathbb{Z} \oplus \mathbb{Z}\{8\}$  is an isomorphism obviously (or by Lemma 5.6).

The restriction of  $\phi^t \circ \phi$  to  $\bigoplus_{i=1}^5 M(SB(A))\{i\}$  is an isomorphism if it is isomorphism in split case by Proposition 3.9. In split case, using Lemma 5.6, it is sufficient to check that the intersection pairing, restricted to the subspace of  $CH^*(X)$  generated by all Schubert classes contained in  $\alpha_1, \dots, \alpha_5$  is nondegenerate. It is the case in codimension  $\neq 4$ , since the classes

□

$$\square\square, \square\square + \square$$

$$\square\square\square, \square\square\square + \square\square, \square\square\square - \square\square + \square$$

$$\square\square\square - \square\square\square, -\square\square\square - \square\square, -\square\square\square + \square\square - \square\square$$

$$-\square\square\square, -\square\square\square + \square\square$$

$$-\square\square\square$$

form the bases of the groups  $CH^i(X)$  of the respective codimensions 1, 2, 3, 5, 6, 7.

As for codimension 4, the elements

$$\square\square\square, -\square\square\square + \square\square + \square, -\square\square$$

span the subspace of  $CH^4(X)$  consisting of cycles which pull-back from  $Gr(3, 6)$ .

The Gram matrix for the elements  $\square\square\square, \square\square, \square$  is

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

This matrix has determinant  $-2$ , which implies that the conditions of Lemma 5.6 hold if 2 is invertible in the coefficient ring.

□

**Remark 5.9.** The complementary summand in the decomposition from Proposition 5.8 corresponds to the vanishing cycle for the embedding of  $X$  into the form of  $SB_2(M_3(A))$ . Indeed, if  $A$  splits, the middle degree group  $CH^4(X)$  is free of rank

4, with 3 generators  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ ,  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$ ,  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  coming from  $Gr(3, 6)$ . The fourth generator is the vanishing cycle  $\delta$  as in Picard-Lefschetz theory. This cycle is characterized by the property that it is sent to 0 under the direct image map. One can prove that if  $A$  splits, there exists a cycle  $Z$  in the class of  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  on  $Gr(3, 6)$  such that the intersection of  $Z$  with  $X$  consists of two components  $\alpha$  and  $\beta$  with multiplicities one, and the vanishing cycle is expressed as

$$\delta = \alpha - \beta + \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} - \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$$

When  $A$  is not split,  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  is still a rational class. However, both  $\alpha$  and  $\beta$  as well as  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  and  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  are not rational (even as elements of Chow groups), therefore it is not clear how to describe  $\delta$  in general.

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